Semester **Real Analysis I** **Time: 3hrs** **Total Marks: 50**
Section I: Answer all and each question is worth 2 Marks **Total Marks 6**

1. Let (a_n) be a Cauchy sequence of real numbers. Suppose there is a subsequence (a_{k_n}) such that $a_{k_n} \to a$. Prove that $a_n \to a$.

2. Determine all continuous functions $f : \mathbb{R} \to \mathbb{R}$ such that $f(x) \in \mathbb{Z}$ for all $x \in \mathbb{R}$

3. Suppose $f : \mathbb{R} \to \mathbb{R}$ is a function satisfying $|f(x) - f(y)| \leq |x - y|^2$ for all $x, y \in \mathbb{R}$. Show that f is constant.

Section II: Answer any 4 and each question is worth 6 Marks **Total Marks 24**

1. Prove that Cauchy sequences are convergent.

2. Let (a_n) be a sequence of real numbers. Let $b_n = |a_n| + a_n$ and $c_n = |a_n| - a_n$ for all $n \geq 1$. Prove that $\sum a_n$ converges absolutely if and only if $\sum a_n$ and $\sum b_n$ converge.

3. Let $f : \mathbb{R} \to \mathbb{R}$ be a function with IVP and $x \in \mathbb{R}$. Suppose $\lim f(x_n) = f(x)$ for any sequence $x_n \to x$ with $(f(x_n))$ is a constant sequence. Prove that f is continuous at x.

4. Prove that a continuous function on $[a, b]$ is uniformly continuous.

5. Let $f : (0, 1) \to \mathbb{R}$ be a differentiable function having a local maximum at $a \in (0, 1)$. Prove that $f'(a) = 0$.

6. Let $f : \mathbb{R} \to \mathbb{R}$ be a differentiable function such $f(x + y) = f(x) + f(y)$ for any $x, y \in [0, 1]$. Prove that there exists $a \in \mathbb{R}$ such that $f(x) = ax$ for all $x \in [0, 1]$.

Section III: Answer any 2 and each question is worth 10 Marks **Total Marks 20**

1. Let (a_n) be a sequence of real numbers.

   (a) If c is a limit point of (a_n), prove that there exists a subsequence (a_{k_n}) such that $a_{k_n} \to c$ and $\liminf a_n \leq c \leq \limsup a_n$.

   (b) Prove that $a_n \to \infty$ if and only if $\liminf a_n = \infty$

2. (a) Let $f\colon \mathbb{R} \to \mathbb{R}$ be a function with IVP. Can f have simple discontinuties? Justify your answer.

   (b) Let $f\colon (a,b) \to \mathbb{R}$ be a strictly increasing continuous function. Prove that there are extended real numbers A and B and a continuous function $\phi\colon (A,B) \to (a,b)$ such that $\phi(f(x)) = x$ for all $x \in (a,b)$.

3. (a) Let $f\colon [a,b] \to \mathbb{R}$ be a differentiable function and $A > 0$ such that $f(a) = 0$ and $|f'(x)| \leq A|f(x)|$ for all $x \in [a,b]$. Prove that $f = 0$ on $[a,b]$.

   (b) Prove Taylor's theorem: Let $f\colon [0,1] \to \mathbb{R}$ be a function such that f' exists and continuous on $[0,1]$ and f'' exists on $(0,1)$. Prove that there is a $t \in (0,1)$ such that $f(1) = f(0) + f'(0) + \frac{f''(t)}{2}$.